MOP 2019 Practice Test 1

USA Blue Group

Thursday, June 6, 2019

Time limit: 4.5 hours. Here we are again! It's always such a pleasure.

B1.1. Let $n \geq 2$ be an integer. There are n^2 marbles, each of which is one of n colors, but not necessarily n of each color. Prove that they may be placed in n boxes with n marbles in each box, such that each box contains marbles of at most two colors.

B1.2. Let ABC be a triangle with circumcircle ω and incenter I. A line ℓ meets the lines AI, BI, CI at points D, E, F respectively, all distinct from A, B, C, I. Prove that the circumcircle of the triangle determined by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ is tangent to ω.

B1.3. Let $n \geq 2018$ be an integer, and let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be pairwise distinct positive integers at most $5n$. Suppose that the fractions

$$\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}$$

form an arithmetic progression (in that order). Prove that all n fractions are equal.

MOP 2019 Practice Test 2

USA Blue Group

Monday, June 10, 2019

Time limit: 4.5 hours. You are filled with determination.

B2.1. Determine the largest number of non-overlapping $1 \times 2 \times 2$ blocks (rotations permitted) that may fit inside a $3 \times 3 \times 3$ cube.

B2.2. Let a_0, a_1, a_2, $\dots$, a_{2018} be a sequence of real numbers such that $a_0 = 0$, $a_1 = 1$, and for every $n \geq 2$ there exists $1 \leq k \leq n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the smallest and largest possible values of $a_{2018} - a_{2017}$.

B2.3. Let O be the circumcenter and Ω be the circumcircle of an acute triangle ABC. Let P be a point on Ω different from A, B, C, as well as their antipodes. Let O_A, O_B, O_C be the circumcenters of $\triangle AOP$, $\triangle BOP$, $\triangle COP$, respectively. Line ℓ_A passes through O_A and is perpendicular to $\overline{BC}$; lines ℓ_B and ℓ_C are defined similarly.

Prove that the circumcircle of the triangle determined by ℓ_A, ℓ_B, ℓ_C is tangent to line OP.

MOP 2019 Practice Test 3

USA Blue Group

Wednesday, June 12, 2019

Time limit: 4.5 hours. Stay determined!

B3.1. Determine whether there exists a sequence $a_1, a_2, \dots$ of nonnegative reals such that

$$a_n + a_{2n} + \cdots = \frac{1}{n}$$

for every positive integer n.

B3.2. Let ABC be an acute triangle with circumcircle Γ and altitudes $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ meeting at H. Let ω be the circumcircle of $\triangle DEF$. Point $S \neq A$ lies on Γ such that $DS = DA$. Line $\overline{AD}$ meets $\overline{EF}$ at Q, and meets ω at $L \neq D$. Point M is chosen such that $\overline{DM}$ is a diameter of ω. Point P lies on $\overline{EF}$ with $\overline{DP} \perp \overline{EF}$. Prove that lines SH, MQ, PL are concurrent.

B3.3. Consider 2018 circles in the plane; suppose each pair of circles intersects twice, and no three circles are concurrent. Let a *vertex* denote an intersection point of two circles (hence there are $2\binom{2018}{2}$ vertices). For each circle, color its vertices red and blue in an alternating fashion. Thus, each vertex is colored twice (once for each circle containing it). If the two colors differ, the vertex is recolored yellow.

The circles partition the plane into several faces (each point not on a circle is in exactly one face), including one unbounded face. Prove that if some circle passes through at least 2061 yellow points, then some face has only yellow vertices.

MOP 2019 Practice Test 4

USA Blue Group

Friday, June 14, 2019

Time limit: 4.5 hours. Good luck next week, thanks for playing.

B4.1. Determine all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

B4.2. For which positive integers n does there exist an $n \times n$ matrix with entries in $\{-1, 0, 1\}$ such that all $2n$ row sums and column sums are pairwise distinct?

B4.3. Let $f\colon \{1, 2, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m+n)$ divides $f(m)+f(n)$ for all positive integers m and n. Prove that there exists a positive integer greater than 1 which divides $f(n)$ for every positive integer n.